THE CORRELATION FUNCTION
IN TWO DIMENSIONAL ISING MODEL
ON THE FINITE SIZE LATTICE. I.

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Abstract

The correlation function of two dimensional Ising model with the nearest neighbours interaction on the finite size lattice with the periodical boundary conditions is derived. The expressions similar to the form factor representation are obtained both for the ferromagnetic and paramagnetic regions of coupling constant. The peculiarities caused by the finite size of the system are discussed. The asymptotic behaviour of the corresponding quantities as functions of size is calculated. Some generalization of the obtained result is conjectured.

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1 Introduction

Since the outstanding result of Montroll, Potts, Ward [1] there appeared many papers devoted to the problem of the spin-spin correlation function in the two dimensional Ising model (IM) with the nearest-neighbour interaction on the infinite size lattice (see e.g. refs. in [2]). The basic IM theory results are collected in the significant book [3]. I can not, of course, to give here any comprehensive list of IM references. Only very few of them which were straight instructive for the preparation of the present paper will be referred to. I think, in particular, that the results and ideas of the paper [4] were very fruitful for the further development of the IM theory. The study of the scaling limit of the IM (see e.g. [4]–[7]) is of interest from the quantum field theory point of view.

The so-called form factor representation [7] for the two dimensional IM correlation function appears to be a crown of this activity:

$$\langle \sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2)\rangle = \mathrm{const} \sum_n g_n(r), \quad r = |\mathbf{r}_1 - \mathbf{r}_2|, \tag{1.1}$$

$$g_n(r) = \frac{1}{n!(2\pi)^n} \int\limits_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\frac{dq_i}{\omega_i} \mathrm{e}^{-r\omega_i} \right) F_n^2[q], \tag{1.2}$$

$$F_n[q] = \prod_{i<j}^{n} \left(\frac{q_i - q_j}{\omega_i + \omega_j} \right), \quad \omega_i = \omega(q_i) = \sqrt{m^2 + q_i^2}. \tag{1.3}$$

In the ferromagnetic region of the coupling parameter the summation in (1.1) is extended over even n, in the paramagnetic — over odd. It is worth of noting that this representation was first evaluated [7] in the framework of the S-matrix approach [8] and then [6] by means of straightforward IM solution. The discovery of the form factor representation was very fruitful for the progress of exactly integrable quantum field theories [9].

The advantage of representation (1.1) consists in that the dynamical aspects of the system are separated from the kinematical ones: form factors squared are integrated over the phase volumes of the n-particle intermediate states. The form factor representation (1.1) clarifies the dynamics of the model but does not answer on the questions about the spectrum, sort and statistics of the particles that form the intermediate and asymptotic states of the system under consideration. The analysis of the model at finite (nonzero) temperature or in the finite volume is necessary for this purpose. One can observe an activity in this field in last years [10]–[16]. These works show in particular that the problem is complicated enough: the authors of refs. [15], [16], for example, call the conjectures of [12], [14] in question. I think that a simple exactly solvable lattice model example would be very useful for the business. Formally the finite temperature (in quantum field theory sense) means the finite size along the temporal axis and periodical boundary condition for boson fields or antiperiodical — for fermion fields. Meanwhile there is no representation which is analogous to the form factor one for the correlation function on the finite size lattice even for the Ising model.

This work, I hope, makes up a deficiency. I have calculated the IM spin-spin correlator on the lattice wrapped on a cylinder by the use of the classic methods of IM theory [3] adapted properly to the case of the finite sizes. The solution is expressed in the form similar

to the form factor expansion (1.1)–(1.3). For reader's convenience I write the result just in the Introduction. If one considers it obvious he would be free of cumbersome mathematical transformations. So, the expression obtained in this work is

$$\langle \sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2) \rangle = \xi \xi_T \mathrm{e}^{-r/\Lambda} \sum_n g_n(r), \tag{1.4}$$

$$g_n(r) = \frac{\mathrm{e}^{-n/\Lambda}}{n!(N)^n} \sum_{[q]}^{(b)} \prod_{i=1}^{n} \left(\frac{\mathrm{e}^{-r\gamma_i - \eta_i}}{\sinh \gamma_i} \right) F_n^2[q], \tag{1.5}$$

$$F_n[q] = \prod_{i<j}^{n} \frac{\sin((q_i - q_j)/2)}{\sinh((\gamma_i + \gamma_j)/2)}. \tag{1.6}$$

More precisely, the representation (1.4)–(1.6) corresponds to the correlation of spins posed on the line parallel to the axis of a cylinder with N sites on its circumference. The values ξ, ξ_T, Λ, γ_i, η_i, in eqs. (1.4)–(1.6) are defined further in Sections 2, 3, 4. Three of them — ξ_T, Λ and η_i — are specific cylinder circumference dependent values: $\ln \xi_T$, Λ^{-1}, η_i vanish if N tends to infinity. The appearance of the summation in (1.5) instead of the integration over phase volume is natural consequence of the size finiteness. It is widely believed that the underlying IM field is the fermion one: for instance the IM partition function is exactly the same as in the free Majorana fermion system. So, the boson spectrum of quasimomentum in Brillouin zone (this is denoted by the upper index in sum (1.5)) is surprising . Nevertheless it follows unambiguously from the calculations. It takes attention also that the lattice form factor (1.6) does not depends explicitly on the cylinder circumference. Note that similar expression for $F_n[q]$ at even n was found by authors of [6] for the infinite lattice case.

In Section 2 the model is formulated and the brief evaluation of the Toeplitz determinant representation for the IM correlation function is given. The corresponding Toeplitz matrix allowing for the finite size lattice is calculated. In Section 3 the lattice form factor representation for the correlation function is derived for the ferromagnetic domain of the coupling parameter and for the paramagnetic domain — in Section 4. In Conclusion I discuss the relationship between the IM correlation function on a cylinder and quantum field Green function at finite temperature or in finite volume. I also comment on some prospect of generalization of the obtained results. In Appendix the factorised representations for the Toeplitz determinants of special form which are used for obtaining lattice form factor expansions are evaluated.

2 The Model

The Ising model with the nearest neighbour interaction on the square $M \times N$ lattice (see. Fig. 1) is defined by the hamiltonian $H[\sigma]$

$$-\beta H[\sigma] = K \sum_{\mathbf{r}} \sigma(\mathbf{r})(\nabla_x + \nabla_y)\sigma(\mathbf{r}), \quad K = \beta J,$$

where J is the coupling parameter, β is the inverse temperature; two dimensional vector $\mathbf{r} = (x, y)$ numerates the lattice sites: $x = 1, 2, \ldots, M, \ y = 1, 2, \ldots, N; \ \nabla_x, \nabla_y$ are the one step shift operators

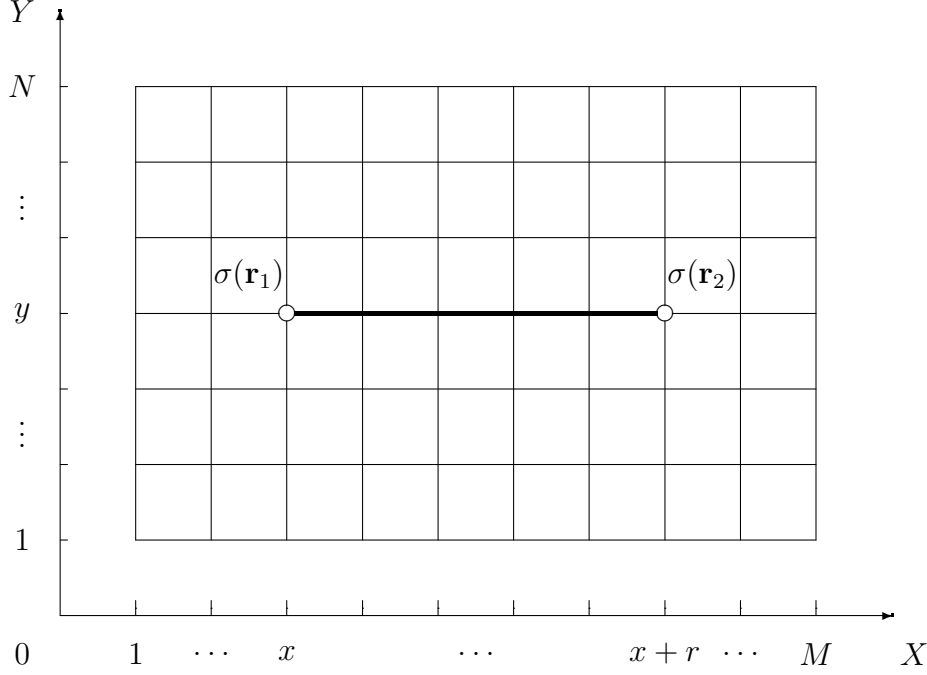$$\nabla_x \sigma(x, y) = \sigma(x + 1, y), \quad \nabla_y \sigma(x, y) = \sigma(x, y + 1).$$

Fig. 1. The positions of the correlating Ising spins on the lattice.

The partition function of the model is

$$Z = \sum_{[\sigma]} e^{-\beta H[\sigma]}, \tag{2.1}$$

the pair correlation function is

$$\langle \sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2) \rangle = Z^{-1} \sum_{[\sigma]} e^{-\beta H[\sigma]} \sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2). \tag{2.2}$$

For the lattice with the periodical boundary conditions (wrapped on a torus) the partition function (2.1) can be expressed [17], [18] through the combination of the four Gauss Grassmann functional integrals with different boundary conditions

$$Z = (2 \cosh^2 K)^{MN} \tfrac{1}{2} (\mathcal{Q}^{(ff)} + \mathcal{Q}^{(bf)} + \mathcal{Q}^{(fb)} - \mathcal{Q}^{(bb)}), \tag{2.3}$$

where

$$\mathcal{Q} = \int d[\psi] e^{S[\psi]}, \qquad d[\psi] = \prod_{\mathbf{r}} d\psi^1(\mathbf{r})d\psi^2(\mathbf{r})d\psi^3(\mathbf{r})d\psi^4(\mathbf{r}).$$

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The action $S[\psi]$ is the anticommuting quadratic form

$$S[\psi] = \tfrac{1}{2} \sum_{\mathbf{r},\mathbf{r}'} \sum_{\nu,\rho=1}^{4} \psi^\nu(\mathbf{r}) D^{\nu\rho}(\mathbf{r},\mathbf{r}') \psi^\rho(\mathbf{r}') = \tfrac{1}{2}(\psi \hat{D} \psi), \tag{2.4}$$

$$\hat{D} = \begin{pmatrix} 0 & 1+t\nabla_x & 1 & 1 \\ -1-t\nabla_{-x} & 0 & -1 & 1 \\ -1 & 1 & 0 & 1+t\nabla_y \\ -1 & -1 & -1-t\nabla_{-y} & 0 \end{pmatrix}, \quad t = \tanh K. \tag{2.5}$$

The indices (f,b) in eq. (2.3) shows the type of the boundary conditions for the shift operators in the matrix (2.5).

"f" – antiperiodical: $(\nabla_x^{(f)})^M = -1$, $(\nabla_y^{(f)})^N = -1$;
   the corresponding quasimomentum runs over halfinteger values in Brillouin zone $(-\pi, \pi)$ (in $2\pi/M$ unites for p_x and $2\pi/N$ – for p_y)

"b" – periodical: $(\nabla_x^{(b)})^M = 1$, $(\nabla_y^{(b)})^N = 1$;
   the corresponding components of quasimomentum runs over integer values.

The correlation function (2.2) can be also presented in terms of the Grassmann functional integrals with the Gauss distribution [18]

$$\langle \sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2) \rangle = t^r \frac{\mathcal{Q}_d^{(ff)} + \mathcal{Q}_d^{(bf)} + \mathcal{Q}_d^{(fb)} - \mathcal{Q}_d^{(bb)}}{\mathcal{Q}^{(ff)} + \mathcal{Q}^{(bf)} + \mathcal{Q}^{(fb)} - \mathcal{Q}^{(bb)}}, \tag{2.6}$$

where $\mathcal{Q}_d$ is the functional integral

$$\mathcal{Q}_d = \int d[\psi] e^{S_d[\psi]}. \tag{2.7}$$

The action with the defect denoted by $S_d[\psi]$ in (2.7) differs from (2.4) in that the parameter t is replaced by t^{-1} on the links along the path which connects the sites $\mathbf{r}_1$ and $\mathbf{r}_2$.

I consider the special case when the sites $\mathbf{r}_1$ and $\mathbf{r}_2$ are situated on the line parallel to the horizontal axis as is shown on Fig. 1:

$$\mathbf{r}_1 = (x,y), \quad \mathbf{r}_2 = (x+r,y), \quad |\mathbf{r}_1 - \mathbf{r}_2| = r.$$

The action with the defect in this case has the form

$$S_d[\psi] = S[\psi] + \Delta S[\psi],$$

where

$$\Delta S[\psi] = (t^{-1} - t) \sum_{x'=1}^{r} \psi^1(x+x'-1,y)\psi^2(x+x',y).$$

The ratio in the r.h.s. of (2.6) simplifies

$$\langle \sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2) \rangle = t^r \frac{\mathcal{Q}_d^{(ff)}}{\mathcal{Q}^{(ff)}} [1 + o(e^{-M/N})], \tag{2.8}$$

if at least one of the lattice sizes is much larger than other, i.e. torus degenerates to cylinder: in our case $M/N \gg 1$. Note, that only the terms with the antiperiodical boundary conditions survive.

After some manipulations with the functional integrals the ratio in the r.h.s. of (2.8) can be transformed to the functional integral over Grassmann field which is defined just on the line connecting the sites $\mathbf{r}_1$ and $\mathbf{r}_2$

$$\langle \sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2) \rangle = \int d[\chi] e^{(\chi^1 A \chi^2)} = \det A, \tag{2.9}$$

$$d[\chi] = \prod_{x=0}^{r-1} d\chi_x^1 \, d\chi_x^2, \quad (\chi^1 A \chi^2) = \sum_{x,x'} \chi_x^1 A_{x,x'} \chi_{x'}^2,$$

$$A_{x,x'} = \frac{1}{MN} \sum_{\mathbf{p}}^{(f)} \frac{e^{ip_x(x-x')}[2t(1+t^2) - (1-t^2)(e^{ip_x} + t^2 e^{-ip_x})]}{(1+t^2)^2 - 2t(1-t^2)(\cos p_x + \cos p_y)}, \tag{2.10}$$

$$x, x' = 0, \, 1, \, \ldots, \, r-1.$$

I recall that the index (f) points the antiperiodicity: the momentum's components run over halfinteger values – fermion spectrum

$$\sum_{p_x}^{(f)} v(p_x) = \sum_{l=1}^{M} v\left(\frac{2\pi}{M}(l + \tfrac{1}{2})\right), \quad \sum_{p_y}^{(f)} v(p_y) = \sum_{l=1}^{N} v\left(\frac{2\pi}{N}(l + \tfrac{1}{2})\right).$$

Now let us rewrite the expression (2.10) in appropriate for the Toeplitz determinant theory manner. From the tabulated formulas [20]

$$\prod_{p_y}^{(f)} 2(\cosh\gamma - \cos p_y) = e^{N\gamma}(1 + e^{-N\gamma})^2, \quad \prod_{p_y}^{(b)} 2(\cosh\gamma - \cos p_y) = e^{N\gamma}(1 - e^{-N\gamma})^2, \tag{2.11}$$

it follows in particular

$$\frac{1}{N}\sum_{p_y}^{(f)} \frac{1}{\cosh\gamma - \cos p_y} = \frac{\tanh(N\gamma/2)}{\sinh\gamma}, \quad \frac{1}{N}\sum_{p_y}^{(b)} \frac{1}{\cosh\gamma - \cos p_y} = \frac{\coth(N\gamma/2)}{\sinh\gamma}. \tag{2.12}$$

At the cylinder case $M \to \infty$, $N = \text{const}$ the summation over p_x turns into the integral

$$\frac{1}{M}\sum_{p_x}^{(f)} v(e^{-ip_x}) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} v(z) + o(e^{-M\mu}). \tag{2.13}$$

Summing up (2.10) over p_y by use of (2.12) and accounting for (2.13) one can obtain for the matrix $A_{x,x'}$

$$A_{x,x'} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-x+x'} A(z), \tag{2.14}$$

where the kernel $A(z)$ is

$$A(z) = \sqrt{\frac{(1 - e^{-\gamma_\pi} z)(1 - e^{-\gamma_0} z^{-1})}{(1 - e^{-\gamma_0} z)(1 - e^{-\gamma_\pi} z^{-1})}} \cdot T(z), \quad z = e^{-ip}, \tag{2.15}$$

and

$$T(z) = \tanh[(N\gamma(p))/2], \tag{2.16}$$

$$\gamma_0 = |2K + \ln t|, \quad \gamma_\pi = 2K - \ln t. \tag{2.17}$$

The function $\gamma(p)$ in (2.16) is defined by the following equation

$$4\sinh^2\frac{\gamma(p)}{2} = \mu^2 + 4\sin^2\frac{p}{2}, \tag{2.18}$$

where

$$\mu^2 = 2(\sqrt{\sinh 2K} - 1/\sqrt{\sinh 2K})^2. \tag{2.19}$$

It follows from eqs. (2.18), (2.19) in particular

$$\gamma(0) = \gamma_0, \quad \gamma(\pi) = \gamma_\pi, \quad \cosh\gamma_\pi - \cosh\gamma_0 = 2.$$

The function $T(z)$ (2.16) tends to unity at the thermodynamic limit $N \to \infty$ (more precise $N\mu \gg 1$), and the kernel $A(z)$ (2.15) turns into the classic IM theory expression. In this limit the value $K = K_c$ is the critical point: the specific heat and correlation length diverge. At the finite N the corresponding singularities are smoothed and there is no phase transition. Nevertheless also in this case the value $K = K_c$ is notable. The calculation technique based on Wiener–Hopf sum equation "distinguishes" the ferro- and paramagnetic regions and the expression for the correlation function at $K > K_c$ differs from that at $K > K_c$. Just in this sense I shall mean further the words "critical point", "phase" and so on. Note that the following equalities take place at the critical point

$$\sinh 2K_c = 1, \quad \mu = 0, \quad \gamma_0 = 0, \quad \gamma_\pi = 2\ln(\sqrt{2} + 1).$$

3 The Ferromagnetic region: $K > K_c$

The dependence of the correlation function (2.9)

$$\langle\sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2)\rangle = \det A^{(r)} \tag{3.1}$$

on the distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$ enters through the matrix dimension denoted by upper index (r) in the r.h.s. of (3.1). The dependence on the cylinder circumference N comes into the function $T(z)$ (2.16) and the dependence on the coupling parameter K is defined by the parameter μ (2.19) in the eq. (2.18) for the function $\gamma(p)$.

The kernel (2.14) of the matrix (2.9) $A^{(r)}$ has to be represented in the factorized form according to the method of the determinant calculation which we use (see (A6), (A7) in the Appendix),

$$A(z) = P(z)Q(z^{-1}),$$

where the functions $P(z)$ and $Q(z)$ are analytic ones inside the $|z| = 1$ circle. It can be formally done by use of the projection operators (A8) that are defined in the Appendix

$$\begin{aligned}
2\ln P(z) &= \ln(1 - e^{-\gamma_\pi}z) - \ln(1 - e^{-\gamma_0}z) + 2\mathcal{P}\ln T(z), \\
2\ln Q(z^{-1}) &= \ln(1 - e^{-\gamma_0}z^{-1}) - \ln(1 - e^{-\gamma_\pi}z^{-1}) + 2\widetilde{\mathcal{P}}\ln T(z).
\end{aligned}$$

Otherwise accounting for the explicit form (2.16) one can express $T(z)$ through the ratio of the products with help of eqs. (2.11)

$$T^2(z) = \frac{\prod\limits_q^{(b)} [\cosh \gamma(p) - \cos q]}{\prod\limits_q^{(f)} [\cosh \gamma(p) - \cos q]}. \tag{3.2}$$

It follows from (2.18)

$$\cosh \gamma(p) - \cos q = 2 + \frac{\mu^2}{2} - \cos q - \cos p = \tfrac{1}{2} e^{\gamma(q)} (1 - e^{-\gamma(q)} z)(1 - e^{-\gamma(q)} z^{-1}),$$

so eq. (3.2) turns into the form

$$T^2(z) = \frac{\prod\limits_q^{(b)} e^{\gamma(q)} (1 - e^{-\gamma(q)} z)(1 - e^{-\gamma(q)} z^{-1})}{\prod\limits_q^{(f)} e^{\gamma(q)} (1 - e^{-\gamma(q)} z)(1 - e^{-\gamma(q)} z^{-1})}. \tag{3.3}$$

Owing to the momentum q runs the entire Brillouin zone all the factors of the r.h.s. (3.3) are double except for the values $q = 0$ and $q = \pi$ in the nominator. So the functions $A(z)$, $P(z)$ and $Q(z^{-1})$ have no branch points rather the simple poles and zeroes:

$$P(z) = \exp\left(\frac{\gamma_0 - \gamma_\pi}{2}\right) \frac{\prod\limits_{0 < q \le \pi}^{(b)} (e^{\gamma(q)} - z)}{\prod\limits_{0 < q < \pi}^{(f)} (e^{\gamma(q)} - z)}, \quad Q(z^{-1}) = \frac{\prod\limits_{0 \le q < \pi}^{(b)} (1 - e^{-\gamma(q)} z^{-1})}{\prod\limits_{0 < q < \pi}^{(f)} (1 - e^{-\gamma(q)} z^{-1})}. \tag{3.4}$$

According to (A25), (A28)

$$\det A^{(r)} = e^{h(r)} \sum_{l=0}^{\infty} g_{2l}(r). \tag{3.5}$$

Let us write the function $h(r)$ as the sum of three terms

$$h(r) = -r/\Lambda + \ln \xi + \ln \xi_T. \tag{3.6}$$

It follows from the eqs. (A23) and (2.15)

$$\Lambda^{-1} = -[\ln P(0) + \ln Q(0)] = -\frac{1}{2\pi i} \oint\limits_{|z|=1} \frac{dz}{z} \ln T(z) = \frac{1}{\pi} \int\limits_0^\pi dp \, \ln \coth(N\gamma(p)/2). \tag{3.7}$$

The parameter Λ has the sense of the coherence length. Its asymptotic behaviour at large N is

$$\Lambda \simeq e^{N\gamma_0} \sqrt{\frac{\pi N}{2 \sinh \gamma_0}}. \tag{3.8}$$

One can see from (3.8) that the coherence length grows very rapidly when the cylinder circumference increases and $\gamma_0 \neq 0$. The lattice can be considered as infinite at $N\gamma_0 \gg \max[\ln(N\gamma_0), \ln(r\gamma_0)]$. It follows also from (A23) that

$$\ln(\xi\xi_T) = \frac{1}{2\pi i} \oint\limits_{|z|=1} dz \ln Q(z^{-1}) \frac{\partial}{\partial z} \ln P(z) = \frac{1}{(2\pi i)^2} \oint\limits_{|z_2|>|z_1|} d(z_1 z_2) \frac{\ln A(z_1) \ln A(z_2)}{(z_2 - z_1)^2}, \qquad (3.9)$$

or separately for the $\ln\xi$ and $\ln\xi_T$

$$\ln\xi = \frac{1}{4} \ln\left[\frac{\sinh\gamma_0 \cdot \sinh\gamma_\pi}{\sinh^2((\gamma_0 + \gamma_\pi)/2)}\right], \qquad (3.10)$$

$$\ln\xi_T = \frac{1}{(2\pi i)^2} \oint\limits_{|z_2|>|z_1|} \frac{d(z_1 z_2)}{(z_2 - z_1)^2} \ln T(z_1) \ln T(z_2). \qquad (3.11)$$

The r.h.s. of the eq. (3.10) does not depend on N and is singular at the critical point

$$\ln\xi \simeq \frac{1}{4} \ln\mu \quad \text{at} \quad \mu \to 0. \qquad (3.12)$$

For the calculation $\ln\xi_T$ let us return to the momentum variables and integrate (3.11) by parts. The result is

$$\ln\xi_T = \frac{N^2}{2\pi^2} \int\limits_0^\pi \frac{dp\,dq\,\gamma'(p)\gamma'(q)}{\sinh(N\gamma(p))\sinh(N\gamma(q))} \ln\left|\frac{\sin((p+q)/2)}{\sin((p-q)/2)}\right|$$

or

$$\ln\xi_T = \frac{N^2}{2\pi^2} \int\limits_{\gamma_0}^{\gamma_\pi} \frac{d\gamma_p\,d\gamma_q}{\sinh(N\gamma(p))\sinh(N\gamma(q))} \ln\left|\frac{\sin((p+q)/2)}{\sin((p-q)/2)}\right|. \qquad (3.13)$$

Differentiating eq. (3.13) with respect to μ one can obtain

$$\frac{\partial}{\partial\mu^2} \ln\xi_T = -\frac{1}{2}[c_1^2(\mu) - c_2^2(\mu)], \qquad (3.14)$$

where

$$c_1(\mu) = \frac{N}{2\pi} \int\limits_{\gamma_0}^{\gamma_\pi} \frac{d\gamma}{\sinh(N\gamma)} \sqrt{\frac{\cosh\gamma_\pi - \cosh\gamma}{\cosh\gamma - \cosh\gamma_0}}, \qquad (3.15)$$

$$c_2(\mu) = \frac{N}{2\pi} \int\limits_{\gamma_0}^{\gamma_\pi} \frac{d\gamma}{\sinh(N\gamma)} \sqrt{\frac{\cosh\gamma - \cosh\gamma_0}{\cosh\gamma_\pi - \cosh\gamma}}.$$

Deriving eqs. (3.14), (3.15) one has to account for that the integrand of (3.13) vanishes on the boundaries of the integration region. Note also that the derivatives of p and q are

$$\frac{\partial p}{\partial\mu^2} = -\frac{1}{2\sin p}, \quad \frac{\partial q}{\partial\mu^2} = -\frac{1}{2\sin q},$$

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that follows from (2.18). For $N\gamma_0 \gg 1$ the asymptotics of $c_1(\mu)$ and $c_2(\mu)$ are

$$c_1(\mu) \simeq \frac{1}{\cosh(N\gamma_0)}\sqrt{\frac{N}{2\pi \sinh \gamma_0}}, \quad c_2(\mu) \simeq \frac{1}{4\cosh(N\gamma_0)}\sqrt{\frac{\sinh \gamma_0}{2\pi N}}. \tag{3.16}$$

Collecting (3.14)–(3.16) one can obtain

$$\ln \xi_T = \int_{\mu}^{\infty} d\nu\, \nu[c_1^2(\nu) - c_2^2(\nu)] \simeq \frac{1}{\pi}e^{-2N\gamma_0}.$$

So, we see that $\xi_T \to 1$ if $N \to \infty$.

The integral (3.13) diverges at the critical point. This means that $\ln \xi_T$ as a function of μ has a singularity. It easy to obtain at $\mu \to 0$

$$c_1(\mu) \simeq 1/2\mu, \quad \frac{\partial \ln \xi_T}{\partial \mu^2} = -\frac{1}{8\mu^2}$$

and consequently

$$\ln \xi_T = -\frac{1}{4}\ln \mu + \text{const.} \tag{3.17}$$

Therefore, one can see from (3.12), (3.17) that the function $\ln(\xi\xi_T)$ has no singularity at $\mu = 0$ if N is finite.

Let now evaluate the expansion in the r.h.s. of (3.5)

$$G_F(r) = \sum_{l=0}^{\infty} g_{2l}(r). \tag{3.18}$$

The coefficients $g_{2l}(r)$ are expressed through the $2l$-multiple contour integrals (A30) as it is shown in the Appendix

$$g_{2l}(r) = \frac{(-1)^l}{l!l!(2\pi i)^{2l}} \oint_{|z_i|<1} \frac{\prod\limits_{i=1}^{2l}(dz_i\, z_i^r) \prod\limits_{i=1}^{l-1}\prod\limits_{j=i+1}^{l}[(z_{2i-1} - z_{2j-1})^2(z_{2i} - z_{2j})^2] \prod\limits_{i=1}^{l} W(z_{2i-1})}{\prod\limits_{i=1}^{l}\prod\limits_{j=1}^{l}(1 - z_{2i-1}z_{2j})^2 \prod\limits_{i=1}^{l} W(z_{2i}^{-1})}. \tag{3.19}$$

The integral (3.19) is defined by the concrete form of the function $W(z) = P(z)/Q(z^{-1})$: in our case this is the eq. (3.4). The integrand in the r.h.s. of (3.19) is analytic at $|z_i| < 1$ except for the singularities which the functions $W(z_i)$ and $W^{-1}(z_i^{-1})$ possess. To separate explicitly the singularities inside the circle $|z| = 1$ these functions ought to be rewritten in the following way

$$W(z) = \frac{P(z)}{Q(z^{-1})} = \sqrt{\frac{(1 - e^{-\gamma_\pi}z)(1 - e^{-\gamma_\pi}z^{-1})}{(1 - e^{-\gamma_0}z)(1 - e^{-\gamma_0}z^{-1})}}\frac{P_T^2(z)}{T(z)}, \tag{3.20}$$

$$W^{-1}(z^{-1}) = \frac{Q(z)}{P(z^{-1})} = \sqrt{\frac{(1 - e^{-\gamma_0}z)(1 - e^{-\gamma_0}z^{-1})}{(1 - e^{-\gamma_\pi}z)(1 - e^{-\gamma_\pi}z^{-1})}}\frac{Q_T^2(z)}{T(z)}. \tag{3.21}$$

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The analytic at $|z| < 1$ functions $P_T(z)$ and $Q_T(z)$ are defined by the equations

$$\ln P_T(z) = \mathcal{P} \ln T(z) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz'}{z' - z} \ln \tanh(N\gamma(p')/2), \tag{3.22}$$

$$\ln Q_T(z^{-1}) = \widetilde{\mathcal{P}} \ln T(z) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz'}{z - z'} \ln \tanh(N\gamma(p')/2), \tag{3.23}$$

$$z' = e^{-ip'}.$$

Note that it follows from (3.7), (3.22), (3.23)

$$\ln P_T(0) = -1/\Lambda, \quad \ln Q_T(0) = 0, \quad P_T(z) = P_T(0) \cdot Q_T(z).$$

With allowance of these eqs. one can obtain instead of (3.20), (3.21)

$$W(z) = e^{(\gamma_0 - \gamma_\pi)/2 - 2/\Lambda}(\cosh \gamma_\pi - \cos p) Q_T^2(z) \frac{\coth(N\gamma(p)/2)}{\sinh \gamma(p)}, \tag{3.24}$$

$$W^{-1}(z^{-1}) = e^{(\gamma_\pi - \gamma_0)/2}(\cosh \gamma_0 - \cos p) Q_T^2(z) \frac{\coth(N\gamma(p)/2)}{\sinh \gamma(p)}, \tag{3.25}$$

$$\frac{\coth(N\gamma(p)/2)}{\sinh \gamma(p)} = \frac{1}{N} \sum_q^{(b)} \frac{1}{\cosh \gamma(q) - \cos p}. \tag{3.26}$$

It is seen from (3.24)–(3.26) that the singularities inside the integration contours of (3.19) are the simple poles at the points

$$z_{(k)} = e^{-\gamma(q_{(k)})}, \quad q_{(k)} = \frac{2\pi k}{N}, \quad k = 1, \ldots, N,$$

corresponding to the boson spectrum of the quasimomentum q. Accounting for

$$\frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} \frac{v(z)}{\cosh \gamma(q) - \cos p} = \frac{v(e^{-\gamma(q)})}{\sinh \gamma(q)},$$

where $v(z)$ is analytic at $|z| < 1$, the integral (3.19) is fulfilled by residua. The result is

$$g_{2l}(r) = \frac{e^{-2l/\Lambda}}{(2l)! N^{2l}} \sum_{[q]}^{(b)} \prod_{i=1}^{2l} \left(\frac{e^{-r\gamma_i - \eta_i}}{\sinh \gamma_i} \right) \mathcal{G}_{2l}[q], \tag{3.27}$$

where

$$\begin{aligned}
\mathcal{G}_{2l}[q] &= C_l^{2l} \prod_{i=1}^{l-1} \prod_{j=i+1}^{l} [(e^{-\gamma_{2i-1}} - e^{-\gamma_{2j-1}})^2 (e^{-\gamma_{2i}} - e^{-\gamma_{2j}})^2] \times \\
&\quad \times \frac{\prod_{i=1}^{l} [(1 + \cos q_{2i-1})(1 - \cos q_{2i})]}{\prod_{i=1}^{2l} e^{\gamma_i} \prod_{i=1}^{l} \prod_{j=1}^{l} (1 - e^{-\gamma_i - \gamma_j})^2},
\end{aligned} \tag{3.28}$$

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$$\gamma_i = \gamma(q_i), \quad \eta_i = \eta(q_i) = -\ln Q_T^2(\mathrm{e}^{-\gamma(q_i)}), \tag{3.29}$$

$$\sum_{[q]}^{(b)} = \sum_{q_1}^{(b)} \sum_{q_2}^{(b)} \cdots \sum_{q_{2l}}^{(b)},$$

$$C_l^{2l} = \frac{(2l)!}{l!l!} \quad - \quad \text{is the binomial coefficient.}$$

Using the equalities

$$\tfrac{1}{2}(\mathrm{e}^{-\gamma_i} - \mathrm{e}^{-\gamma_j})(1 - \mathrm{e}^{\gamma_i+\gamma_j}) = \cosh\gamma_i - \cosh\gamma_j = \cos q_j - \cos q_i,$$

the eq. (3.28) can be transformed to

$$\mathcal{G}_{2l}[q] = \frac{V_{2l}[q]}{\prod\limits_{i<j}^{2l} \sinh^2((\gamma_i+\gamma_j)/2)},$$

where

$$V_{2l}[q] = 2^{-2l^2} C_l^{2l} \prod_{i=1}^{l-1}\prod_{j=i+1}^{l} [(\cos q_{2i-1} - \cos q_{2j-1})^2(\cos q_{2i} - \cos q_{2j})^2] \times$$

$$\times \prod_{i=1}^{l}(1+\cos q_{2i-1})(1-\cos q_{2i}). \tag{3.30}$$

The expression (3.30) is not symmetric with respect to changes the summation variables q_i with even subscripts by that with odd subscripts. The summation extracts the symmetric part of the function $V_{2l}[q]$ which is denoted by $V_{2l}^{\mathrm{sim}}[q]$

$$V_{2l}^{\mathrm{sim}}[q] = \frac{1}{C_l^{2l}} \sum_{\mathrm{P}} V_{2l}[q],$$

where the sum over all permutations of q_{2i-1} and q_{2j} (theirs number is just C_l^{2l}) is denoted by $\sum\limits_{\mathrm{P}}$. The equality

$$V_{2l}^{\mathrm{sim}}[q] = U_{2l}^{\mathrm{even}}[q], \tag{3.31}$$

is the crucial for the final result. Here the even (with respect to $q_i \leftrightarrow -q_i$) part of the product

$$U_{2l}[q] = \prod_{i<j}^{2l} \sin^2\left(\frac{q_i - q_j}{2}\right),$$

is denoted by $U_{2l}^{\mathrm{even}}[q]$

$$U_{2l}^{\mathrm{even}}[q] = 2^{-2l} \sum_{[\sigma=\pm 1]} \prod_{i<j}^{2l} \sin^2\left(\frac{\sigma_i q_i - \sigma_j q_j}{2}\right)$$

Consequently the function $V_{2l}[q]$ under summation (3.27) can be replaced by the function $U_{2l}[q]$. This makes it possible to present the coefficient $g_{2l}(r)$ in the form similar to that of eq. (1.2)

$$n = 2l,$$

$$g_n(r) = \frac{e^{-n/\Lambda}}{n! N^n} \sum_{[q]}^{(b)} \prod_{i=1}^{n} \left(\frac{e^{-r\gamma_i - \eta_i}}{\sinh \gamma_i} \right) F_n^2[q_i], \qquad (3.32)$$

where $F_n[q_i]$ is the lattice analog of the form factor (1.3)

$$F_n[q_i] = \prod_{i<j}^{n} \left[\frac{\sin((q_i - q_j)/2)}{\sinh((\gamma_i + \gamma_j)/2)} \right], \qquad F_0[q] = 1. \qquad (3.33)$$

I emphasize once more that the momentum spectrum over which the summation (3.32) is extended (the intermediate states summing up) occurs to be boson one contrary to the fermion spectrum by which the initial values were defined, in particular the matrix $A_{x,x'}$ (2.10).

The function $\eta_i = \eta(q_i)$ that is contained in (3.32), decreases rapidly when the cylinder circumference grows. It follows from its definition (3.29)

$$\eta(q) = -\frac{1}{\pi i} \oint_{|z|=1} \frac{dz \ln T(z)}{e^{\gamma(q)} - z} = \frac{1}{\pi} \int_0^{\pi} \frac{dp(1 - e^{-\gamma(q)} \cos p)}{\cosh \gamma(q) - \cos p} \ln \coth(N\gamma(p)/2). \qquad (3.34)$$

One can find at $N\gamma_0 \gg 1$

$$\eta(q) \simeq \frac{4}{e^{\gamma(q)} - 1} \frac{e^{-N\gamma_0}}{\sqrt{2\pi N \sinh \gamma_0}},$$

and $\eta(q) \to 0$ if $N \to \infty$.

4 The Paramagnetic region: $\mathbf{K < K_c}$

The correlation function is defined uniform through the determinant $\det A^{(r)}$ in the whole region of parameters (including $K < K_c$). But the method of its calculation is to be modified because the Wiener–Hopf sum equation technique, which is used, demands the Toeplitz matrix kernel of the appropriate form. Both the kernel and the logarithm of the kernel have to satisfy the Loran expansion condition. Meanwhile, the matrix kernel (2.10) is reforming compare to the ferromagnetic case (2.15), when K goes across K_c

$$A(z) = z^{-1} B(z),$$

where

$$B(z) = \sqrt{\frac{(1 - e^{-\gamma_0} z)(1 - e^{-\gamma_\pi} z)}{(1 - e^{-\gamma_0} z^{-1})(1 - e^{-\gamma_\pi} z^{-1})}} T(z), \qquad (4.1)$$

so, rather the function $B(z)$ than $A(z)$ possesses appropriate analytical properties. The matrix $A^{(r)}$ now has the following form

$$A_{x,x'} = -\frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-x+x'-1} B(z). \tag{4.2}$$

Factorizing the kernel (4.1)

$$B(z) = P(z) \cdot Q(z^{-1}),$$

where the functions $P(z)$ and $Q(z)$ are analytic inside the circle $|z| < \mathrm{e}^{\gamma_0}$ one can obtain

$$P(z) = \exp\left(-\frac{\gamma_0 + \gamma_\pi}{2}\right) \frac{\prod\limits_{0 \le q \le \pi}^{(b)} (\mathrm{e}^{\gamma(q)} - z)}{\prod\limits_{0 < q < \pi}^{(f)} (\mathrm{e}^{\gamma(q)} - z)}, \qquad Q(z^{-1}) = \frac{\prod\limits_{0 < q < \pi}^{(b)} (1 - \mathrm{e}^{-\gamma(q)} z^{-1})}{\prod\limits_{0 < q < \pi}^{(f)} (1 - \mathrm{e}^{-\gamma(q)} z^{-1})}. \tag{4.3}$$

Comparing with the factorized representation in the ferromagnetic case one can see that the corresponding products (3.4) and (4.3) differ from each other by one term: the factor $(\mathrm{e}^{\gamma_0} - z)$ is appeared in $P(z)$ and the factor $(1 - \mathrm{e}^{-\gamma_0} z^{-1})$ is disappeared in $Q(z^{-1})$. It follows from (4.3)

$$P(0) = P_T(0) = \mathrm{e}^{-1/\Lambda}, \qquad Q(0) = Q_T(0) = 1,$$

where the functions $P_T(z)$ and $Q_T(z)$ are the same as in the ferromagnetic case (3.22), (3.23).

The matrix (4.2) has the structure (A31) considered in the Appendix. Therefore its determinant can be represented according to the eqs. (A32)–(A35) by the following way

$$\det A^{(r)} = \mathrm{e}^{h(r+1)} F(r), \tag{4.4}$$

$$F(r) = \sum_{l=0}^{\infty} f_{2l+1}(r).$$

The function $h(r + 1)$ is given by eq. (A23). Writing it similarly to (3.6) and allowing for the difference between the definitions (4.3) and (3.4) for the corresponding functions $P(z)$ and $Q(z)$ one can obtain

$$h(r + 1) = -(r + 1)/\Lambda + \ln(\xi \xi_T) + \ln(1 - \mathrm{e}^{-\gamma_0 - \gamma_\pi}), \tag{4.5}$$

where the values Λ, ξ and ξ_T are the same as in the ferromagnetic phase (3.7), (3.10) and (3.11). With the allowance (4.5) the eq. (4.4) can be rewritten as follows

$$\det A^{(r)} = \sinh((\gamma_0 + \gamma_\pi)/2) \mathrm{e}^{h(r)} G_P(r), \tag{4.6}$$

where

$$G_P(r) = 2\mathrm{e}^{-1/\Lambda - (\gamma_0 + \gamma_\pi)/2} F(r) = \sum_{l=0}^{\infty} g_{2l+1}(r), \tag{4.7}$$

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and for the coefficients $g_{2l+1}(r)$ in the expansion (4.7) one obtains from (A35)

$$g_{2l+1}(r) = 2e^{-1/\Lambda - (\gamma_0 + \gamma_\pi)/2} \frac{(-1)^l}{l!(l+1)!} (2\pi i)^{2l+1} \oint_{|z_i|<1} \prod_{i=1}^{2l+1} (dz_i\, z_i^r) \times$$

$$\times \frac{\displaystyle\prod_{i=1}^{l-1}\prod_{j=i+1}^{l}(z_{2i}-z_{2j})^2 \prod_{i=1}^{l}\prod_{j=i+1}^{l+1}(z_{2i-1}-z_{2j-1})^2}{\displaystyle\prod_{i=1}^{l}\prod_{j=0}^{l}(1-z_{2i}z_{2j+1})^2} \frac{\displaystyle\prod_{i=1}^{l}(z_{2i}W(z_{2i}))}{\displaystyle\prod_{i=0}^{l}(z_{2i+1}W(z_{2i+1}^{-1}))} . \qquad (4.8)$$

The functions $W(z) = P(z)/Q(z^{-1})$ and $W^{-1}(z^{-1}) = Q(z)/P(z^{-1})$ entering the integrals (4.8) have the following form in this case

$$W(z) = 2e^{-2/\Lambda - (\gamma_0 + \gamma_\pi)/2} Q_T^2(z) \sinh^2 \gamma(p) \frac{1}{N} \sum_q^{(b)} \frac{1}{\cosh\gamma(q) - \cos p}, \qquad (4.9)$$

$$W^{-1}(z^{-1}) = \frac{1}{2} e^{(\gamma_0+\gamma_\pi)/2} Q_T^2(z) \frac{1}{N} \sum_q^{(b)} \frac{1}{\cosh\gamma(q) - \cos p}. \qquad (4.10)$$

It follows from (4.9), (4.10) that the integrand in (4.8) has only the simple poles inside the integration contour at the points

$$z_{(k)} = e^{-\gamma(q_{(k)})}, \quad q_{(k)} = \frac{2\pi k}{N}, \quad k = 1, 2, \ldots, N;$$

similar to the previous case. At these points, obviously,

$$\cos p = \cosh\gamma(q), \quad \cosh\gamma(p) = \cos q, \quad \sinh^2\gamma(p) = -\sin^2 q.$$

So, the integration contours in (4.8) may be squeezed and the coefficient $g_{2l+1}(r)$ is expressed through the sum of residua

$$g_{2l+1}(r) = \frac{e^{-(2l+1)/\Lambda}}{(2l+1)! N^{2l+1}} \sum_{[q]}^{(b)} \prod_{i=1}^{2l+1} \left(\frac{e^{-r\gamma_i - \eta_i}}{\sinh\gamma_i}\right) \frac{V_{2l+1}[q]}{\prod\limits_{i<j}^{2l+1} \sinh^2((\gamma_i+\gamma_j)/2)}, \qquad (4.11)$$

where

$$V_{2l+1}[q] = C_l^{2l+1} 2^{-2l(l+1)} \prod_{i=1}^{l-1}\prod_{j=i+1}^{l} (\cos q_{2i} - \cos q_{2j})^2 \times$$

$$\times \prod_{i=1}^{l}\prod_{j=i+1}^{l+1} (\cos q_{2i-1} - \cos q_{2j-1})^2 \prod_{i=1}^{l} \sin^2 q_{2i}. \qquad (4.12)$$

The expression (4.12) differs from the corresponding ferromagnetic one (3.30). But it occurs that also in the paramagnetic case the symmetric part of the function $V_{2l+1}[q]$ – with respect to change of each variable with even subscript q_{2i} by each variable with odd subscript q_{2j-1}

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(the number of the permutations is C_l^{2l+1}) – coincides with the even part of the function $U_{2l+1}[q]$

$$U_{2l+1}[q] = \prod_{i<j}^{2l+1} \sin^2\left(\frac{q_i - q_j}{2}\right).$$

The equality analogous to the eq. (3.31) takes place

$$V_{2l+1}^{\mathrm{sim}}[q] = U_{2l+1}^{\mathrm{even}}[q],$$

where

$$V_{2l+1}^{\mathrm{sim}}[q] = \frac{1}{C_l^{2l+1}} \sum_{\mathrm{P}} V_{2l+1}[q],$$

$$U_{2l+1}^{\mathrm{even}}[q] = 2^{-(2l+1)} \sum_{[\sigma=\pm1]} \prod_{i<j}^{2l+1} \sin^2\left(\frac{\sigma_i q_i - \sigma_j q_j}{2}\right).$$

Consequently the function $V_{2l+1}[q]$ can be replaced by the function $U_{2l+1}[q]$ under the summation in the r.h.s. of eq. (4.11). We obtain eventually

$$n = 2l + 1,$$

$$g_n(r) = \frac{\mathrm{e}^{-n/\Lambda}}{n! N^n} \sum_{[q]}^{(b)} \prod_{i=1}^{n} \left(\frac{\mathrm{e}^{-r\gamma_i - \eta_i}}{\sinh \gamma_i}\right) F_n^2[q], \tag{4.13}$$

$$F_n[q] = \prod_{i<j}^{n} \frac{\sin((q_i - q_j)/2)}{\sinh((\gamma_i + \gamma_j)/2)}, \quad F_1[q] = 1. \tag{4.14}$$

The functions $\gamma_i = \gamma(q_i)$ and $\eta_i = \eta(q_i)$ are defined by the eq. (3.29).

The form of the coefficients $g_n(r)$ and of the function $F_n[q]$ is surprisingly the same both the ferro- and paramagnetic cases. In the ferromagnetic case the correlation function expansion is extended over even n and for the paramagnetic case – over odd n

$$\langle \sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2)\rangle = (\xi \cdot \xi_T)\mathrm{e}^{-r/\Lambda} \sum_{l=0}^{\infty} g_{2l}(r), \qquad K > K_c; \tag{4.15}$$

$$\langle \sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2)\rangle = \sinh\left(\frac{\gamma_0 + \gamma_\pi}{2}\right)(\xi \cdot \xi_T)\mathrm{e}^{-r/\Lambda} \sum_{l=0}^{\infty} g_{2l+1}(r), \quad K < K_c. \tag{4.16}$$

Notice that the even values of the circumference N were assumed in the factorization procedure for the functions $P(z)$ and $Q(z^{-1})$ (see eqs. (3.4) and (4.3)). For the odd N the Brillouin zone does not contain the $q = \pi$ point in the boson case. On the contrary this value appears in the fermion spectrum. It is not difficult to account for this detail, it is irrelevant for the final result.

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5 Conclusion

It should be observed first of all that, although the representations (4.15), (4.16) for the correlation function are obtained for different regions of the coupling parameter ($K > K_c$ and $K < K_c$ respectively), both of them are valid and equal to each other for all values of K at any finite cylinder circumference N. For the finite N the expansions (4.15), (4.16) are not series rather finite sums due to the form factor $F_n[q]$ is identically zero for $n > N$. Only the thermodynamic limit $N \to \infty$ splits the domain of their validity.

The obtained lattice form factor representations possess one beautiful feature: they can be checked numerically or analytically. For small values of N the result may be computed in other way, for example with the help of the transfer matrix technique. In particular at $N = 1$ we have the one dimensional Ising chain. The elementary calculation gives

$$\langle \sigma(0)\sigma(r) \rangle = t^r. \tag{5.1}$$

One can see after simple transformations that both the representations (4.15) and (4.16) coincide with (5.1).

Apart from its intrinsic interest the Ising model deserves scrutiny because of its relevance to nonperturbative explorations in quantum field theory. Mutual correspondence between field model and spin lattice system in scaling regime implies in our case the limit $K \to K_c$, $(N, M, r) \to \infty$ at $(N\mu, M\mu, r\mu) = $ const. Mysterious process of thermodynamic or scaling limit may be observed in detail proceeding from the explicit expressions obtained in this work.

The IM correlation function $\langle \sigma(\mathbf{r})\sigma(\mathbf{r}') \rangle$ in scaling limit corresponds to the quantum field model two point Green function $G(x, t; L, \beta)$. The variables of the Green function are assumed to be scaled on mass: $m = 1$, $x = \mu|r_x - r_x'|$ is the spatial distance, $t = \mu|r_y - r_y'|$ is the temporal one. $L = \mu M$ is the volume, $\beta = \mu N$ is the temperature. The boundary conditions along the temporal coordinate are quite definite: periodic for the boson field, antiperiodic for the fermion field. This is the issue of the appropriate formulation of the quantum field theory at finite temperature. On the contrary, the spatial boundary condition may be arbitrary. In any case the presence of boundaries breaks down explicitly the Lorentz invariance (the rotational invariance in Euclidian metric case). The spatial and temporal coordinates are not equivalent in general. In this sense the configuration of correlating spins considered in this paper corresponds to the equal time Green function $G(x, 0; \infty, \beta)$ in the infinite volume at finite temperature. With some stipulations this function may be viewed also as that at zero temperature in finite volume $\widetilde{L} = \mu N$ (with periodicity in spatial coordinate). One has to keep in mind in addition that this is not equal time rather equal site Green function $G(0, \widetilde{t}; \widetilde{L}, \infty)$. It is necessary to consider the correlating spins placed on a cylinder circumference to compute the equal time Green function in finite volume and zero temperature. The corresponding problem may be solved by the method used in this work but with the Toeplitz determinant technique being properly modified.

Meanwhile, the representation (4.13)–(4.16) for the IM correlation function has such transparent structure that the conjecture for the case of general displacement of correlating

spins suggests itself. The simplest one is

$$\langle \sigma(0)\sigma(r_x, r_y)\rangle = (\xi \cdot \xi_T) e^{-r_x/\Lambda} \sum_n g_n(r_x, r_y), \tag{5.2}$$

$$g_n(r_x, r_y) = \frac{e^{-n/\Lambda}}{n! N^n} \sum_{[q]}^{(b)} \prod_{j=1}^{n} \left(\frac{e^{-r_x \gamma_j - i r_y q_j - \eta_j}}{\sinh \gamma_j} \right) F_n^2[q]. \tag{5.3}$$

It is amazing that this generalization is in exact agreement with the transfer matrix results for the N-rows Ising chains. We have checked it analytically for $N = 2, 3, 4$ and numerically for $N = 5, 6$.

Appendix

The elements of the Toeplitz matrix arranged along the diagonals are equal

$$A_{x,x'} = A_k \quad \text{for} \quad k = x - x' = 0, \pm 1, \ldots, \pm r - 1, \tag{A1}$$

$$A^{(r)} = \begin{pmatrix} A_0 & A_{-1} & A_{-2} & \cdots & A_{-r+1} \\ A_1 & A_0 & A_{-1} & \cdots & A_{-r+2} \\ A_2 & A_1 & A_0 & \cdots & A_{-r+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{r-1} & A_{r-2} & A_{r-3} & \cdots & A_0 \end{pmatrix}. \tag{A2}$$

The superscript in $A^{(r)}$ shows the matrix dimension. If there is the function of the complex variable $A(z)$ analytical in the ring $\alpha < |z| < \alpha^{-1}$ ($\alpha < 1$) such that the coefficients A_k of its Loran series

$$A(z) = \sum_{k=-\infty}^{\infty} A_k z^k \tag{A3}$$

coincide with the matrix element (A2) then the matrix (A1) can be represented in the form

$$A_{x,x'} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-x+x'} A(z). \tag{A4}$$

The function $A(z)$ in the integrand of (A4) is called the kernel of the Toeplitz matrix $A^{(r)}$.

If the inverse matrix is known then the calculation of the determinant reduces to calculation of trace with help of the identity

$$\ln \det A^{(r)}(\lambda) = \int d\lambda \, \mathrm{Sp} \left[(A^{(r)}(\lambda))^{-1} \frac{\partial}{\partial \lambda} A^{(r)}(\lambda) \right] + \text{const}, \tag{A5}$$

where λ is a parameter of which the matrix elements depend on explicitly.

The inversion procedure for the Toeplitz matrix simplifies if both the kernel (A3) and $\ln A(z)$ may be expanded in Loran series. If so, the kernel $A(z)$ can be expressed in factorized form

$$A(z) = P(z) \cdot Q(z^{-1}). \tag{A6}$$

The functions $P(z)$ and $Q(z)$ are analytic inside the circle $|z| \leq \alpha^{-1}$:

$$\ln P(z) = \mathcal{P} \ln A(z), \quad \ln Q(z^{-1}) = \widetilde{\mathcal{P}} \ln A(z). \tag{A7}$$

The projection operators $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ extract the analytic part of the given function $v(z)$ inside and outside the circle $|z| = 1$ correspondingly

$$\mathcal{P}^2 = \mathcal{P}, \quad \widetilde{\mathcal{P}}^2 = \widetilde{\mathcal{P}}, \quad \mathcal{P}\widetilde{\mathcal{P}} = \widetilde{\mathcal{P}}\mathcal{P} = 0, \quad \mathcal{P} + \widetilde{\mathcal{P}} = 1,$$

$$\mathcal{P}v(z) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{d\xi v(\xi)}{\xi - z}, \quad \widetilde{\mathcal{P}}v(z) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{d\xi v(\xi)}{z - \xi}. \tag{A8}$$

The problem of the Toeplitz matrix inversion can be reduced to solving the system of the Wienner-Hopf sum equations [3]. In terms of the projection operators (A8) the matrix elements of the inverse matrix $(A^{(r)})_{x,x'}^{-1}$ are

$$(A^{(r)})_{x,x'}^{-1} = \mathcal{P}\frac{z^{r-x}}{Q(z^{-1})}\widetilde{\mathcal{P}}w^{-1}(z)[1 - \widetilde{\mathcal{P}}w(z)\mathcal{P}w^{-1}(z)]^{-1}\mathcal{P}\frac{z^{x'}}{Q(z^{-1})}\bigg|_{z=0}, \tag{A9}$$

$$(A^{(r)})_{x,x'}^{-1} = \mathcal{P}\frac{z^{-x}}{P(z)}\mathcal{P}w(z)[1 - \mathcal{P}w^{-1}(z)\widetilde{\mathcal{P}}w(z)]^{-1}\widetilde{\mathcal{P}}\frac{z^{x'-r}}{P(z)}\bigg|_{z=0}, \tag{A10}$$

$$(A^{(r)})_{x,x'}^{-1} = \mathcal{P}\frac{z^{-x}}{P(z)}\left\{1 - \mathcal{P}[1 - w(z)\mathcal{P}w^{-1}(z)\widetilde{\mathcal{P}}]^{-1}w(z)\mathcal{P}w^{-1}(z)\right\}\mathcal{P}\frac{z^{x'}}{Q(z^{-1})}\bigg|_{z=0}, \tag{A11}$$

where

$$w(z) = z^r W(z), \quad W(z) = \frac{P(z)}{Q(z^{-1})}. \tag{A12}$$

Each of the alternative but equivalent representations (A9)–(A11) for the inverse matrix $(A^{(r)})_{x,x'}^{-1}$ are preferable for the calculation of different matrix elements. In particular, the eq. (A9) is suitable for the calculation of the element $(A^{(r)})_{r-1,0}^{-1}$, the eq. (A11) – for the element $(A^{(r)})_{0,0}^{-1}$:

$$(A^{(r)})_{r-1,0}^{-1} = \frac{1}{Q^2(0)}\left\{z\widetilde{\mathcal{P}}w^{-1}(z)[1 - \widetilde{\mathcal{P}}w(z)\mathcal{P}w^{-1}(z)]^{-1}\right\}\bigg|_{z\to\infty}, \tag{A13}$$

$$(A^{(r)})_{0,0}^{-1} = \frac{1}{P(0)Q(0)}\left\{1 - \mathcal{P}[1 - w(z)\mathcal{P}w^{-1}(z)\widetilde{\mathcal{P}}]^{-1}w(z)\mathcal{P}w^{-1}(z)\right\}\bigg|_{z=0}. \tag{A14}$$

Expanding the inverse operators that are contained in the r.h.s. of (A13) and (A14) into geometrical progression

$$[1 - \widetilde{\mathcal{P}}w(z)\mathcal{P}w^{-1}(z)]^{-1} = \sum_{l=0}^{\infty}[\widetilde{\mathcal{P}}w(z)\mathcal{P}w^{-1}(z)]^l,$$

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$$[1 - w(z)\mathcal{P}w^{-1}(z)\widetilde{\mathcal{P}}]^{-1} = \sum_{l=0}^{\infty}[w(z)\mathcal{P}w^{-1}(z)\widetilde{\mathcal{P}}]^{l}$$

and using the integral representations for the projection operators $\mathcal{P}$, $\widetilde{\mathcal{P}}$ (A8) one can obtain for (A13) with allowance (A12)

$$(A^{(r)})_{r-1,0}^{-1} = \frac{1}{Q^2(0)}\sum_{l=0}^{\infty} a_{2l+1}(r), \qquad (A15)$$

$$a_{2l+1}(r) = \frac{1}{(2\pi i)^{2l+1}} \oint_{|z_i|<1} \frac{\prod\limits_{i=1}^{2l+1}(dz_i\, z_i^r)}{z_1 z_{2l+1}\prod\limits_{i=1}^{2l}(1 - z_i z_{i+1})} \frac{\prod\limits_{i=1}^{l}W(z_{2i})}{\prod\limits_{i=0}^{l}W(z_{2i+1}^{-1})}. \qquad (A16)$$

In particular, the coefficients $a_1(r)$ and $a_3(r)$ are

$$a_1(r) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz\, z^{r-2}}{W(z^{-1})},$$

$$a_3(r) = \frac{1}{(2\pi i)^3} \oint_{|z_i|<1} \frac{d(z_1 z_2 z_3)\,(z_1 z_2 z_3)^r}{z_1 z_3(1 - z_1 z_2)(1 - z_2 z_3)} \frac{W(z_2)}{W(z_1^{-1})W(z_3^{-1})}.$$

Analogously one can obtain for the eq. (A14)

$$(A^{(r)})_{0,0}^{-1} = \frac{1}{P(0)Q(0)}\left(1 - \sum_{l=1}^{\infty} a_{2l}(r)\right), \qquad (A17)$$

$$a_{2l}(r) = \frac{1}{(2\pi i)^{2l}} \oint_{|z_i|<1} \frac{\prod\limits_{i=1}^{2l}(dz_i\, z_i^r)}{z_1 z_{2l}\prod\limits_{i=1}^{2l-1}(1 - z_i z_{i+1})} \frac{\prod\limits_{i=1}^{l}W(z_{2i-1})}{\prod\limits_{i=1}^{l}W(z_{2i}^{-1})} \qquad (A18)$$

and, in particular,

$$a_2(r) = \frac{1}{(2\pi i)^2} \oint_{|z_i|<1} \frac{d(z_1 z_2)(z_1 z_2)^r}{z_1 z_2(1 - z_1 z_2)} \frac{W(z_1)}{W(z_2^{-1})},$$

$$a_4(r) = \frac{1}{(2\pi i)^4} \oint_{|z_i|<1} \frac{d(z_1 z_2 z_3 z_4)(z_1 z_2 z_3 z_4)^r}{z_1 z_4(1 - z_1 z_2)(1 - z_2 z_3)(1 - z_3 z_4)} \frac{W(z_1)W(z_3)}{W(z_2^{-1})W(z_4^{-1})}.$$

Consider now the auxiliary Toeplitz matrix $A^{(r)}(\lambda)$ depending on the parameter λ,

$$A_{x,x'}(\lambda) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-x+x'} P(z)Q^{\lambda}(z^{-1}), \quad 0 \le \lambda \le 1, \qquad (A19)$$

where the functions $P(z)$ and $Q(z)$ are the same as in the kernel of the matrix $A^{(r)}$. It is seen from the definition (A19) that at $\lambda = 1$

$$A_{x,x'}(1) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-x+x'} P(z) Q(z^{-1}) = A_{x,x'},$$

and at $\lambda = 0$

$$A_{x,x'}(0) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-x+x'} P(z). \tag{A20}$$

The matrix (A20) is the triangular one: all its elements to the right of the main diagonal are equal to zero

$$A_{x,x}(0) = P(0), \quad A_{x,x'}(0) = 0 \text{ if } x' > x.$$

Therefore

$$\det A^{(r)}(0) = P^r(0). \tag{A21}$$

The integration constant in (A5) is defined with help of (A21).

Let us select the contribution $h(r)$ in $\ln \det A^{(r)}$ which does not vanish when the matrix dimension grows

$$\ln \det A^{(r)} = h(r) + \Delta h(r), \quad \Delta h(r) \to 0 \text{ if } r \to \infty.$$

It is not difficult to observe that only the first term of the inverse operator expansion in (A10) contributes in $h(r)$. This fact results in drastic simplification of corresponding computations. The inverse matrix (A10) is

$$
\begin{aligned}
(A^{(r)}(\lambda))^{-1}_{x,x'} &= \mathcal{P} \frac{z^{-x}}{P(z)} \mathcal{P} \frac{z^r P(z)}{Q^\lambda(z^{-1})} \widetilde{\mathcal{P}} \frac{z^{x'r}}{P(z)} = \\
&= \frac{1}{(2\pi i)^3} \oint_{|z_2|>|z_{1,3}|} \frac{d(z_1 z_2 z_3) z_1^{-x} (z_2/z_3) z_3^{-x'}}{z_1(z_2 - z_1)(z_2 - z_3)} \frac{P(z_2)}{P(z_1) Q^\lambda(z_2^{-1}) P(z_3)}.
\end{aligned}
$$

The derivative of the matrix (A19) is

$$\frac{\partial}{\partial \lambda} A^{(r)}_{x,x'}(\lambda) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-x+x'} P(z) Q^\lambda(z^{-1}) \ln Q(z^{-1}),$$

and finally one has for the trace in (A5)

$$
\begin{aligned}
\mathrm{Sp}\left[(A^{(r)}(\lambda))^{-1} \frac{\partial}{\partial \lambda} A^{(r)}(\lambda)\right] &= \frac{1}{(2\pi i)^4} \oint_{|z_{2i}|>|z_{2j-1}|} \prod_{i=1}^4 dz_i \times \\
\times \frac{[1 - (z_4/z_1)^r][1 - (z_3/z_4)^r](z_2/z_3)^r}{(z_2 - z_1)(z_2 - z_3)(z_4 - z_1)(z_4 - z_3)} &\cdot \frac{P(z_2) P(z_4) Q^\lambda(z_4^{-1}) \ln Q(z_4^{-1})}{P(z_1) Q^\lambda(z_2^{-1}) P(z_3)} = \\
= r \ln Q(0) + \frac{1}{2\pi i} \oint_{|z|=1} dz \ln Q(z^{-1}) \frac{\partial}{\partial z} \ln P(z) &+ o(r^{-1}). \tag{A22}
\end{aligned}
$$

It is seen from (A22) that nonvanishing contribution in the trace occurs to be independent on the parameter λ. Therefore, with allowance (A21) we obtain

$$h(r) = r[\ln P(0) + \ln Q(0)] + \frac{1}{2\pi i} \oint_{|z|=1} dz \ln Q(z^{-1}) \frac{\partial}{\partial z} \ln P(z). \tag{A23}$$

For the calculation of $\Delta h(r)$ the eq. (A5) may be used by keeping the next terms in the expansion of (A10). But it is more convenient for this purpose to exploit the specific property of the Toeplitz matrix i.e.

$$(A^{(r+1)})_{0,0}^{-1} = \frac{\det A^{(r)}}{\det A^{(r+1)}}.$$

Therefore

$$\det A^{(r)} = \det A^{(r+k)} \prod_{s=r+1}^{r+k} (A^{(s)})_{0,0}^{-1},$$

and after substitution (A17) we obtain

$$\det A^{(r)} = e^{h(r+k)+\Delta h(r+k)} [P(0)Q(0)]^{-k} \prod_{s=r+1}^{r+k} \left(1 - \sum_{l=1}^{\infty} a_{2l}(s)\right). \tag{A24}$$

It follows from (A23) that

$$e^{h(r+k)} = [P(0)Q(0)]^k e^{h(r)},$$

so, the factors $P(0)$ and $Q(0)$ in (A24) cancel. Due to this fact the limit $k \to \infty$ in (A24) can be taken

$$\det A^{(r)} = G(r) \cdot e^{h(r)}, \tag{A25}$$

where

$$G(r) = \prod_{s=r+1}^{\infty} \left(1 - \sum_{l=1}^{\infty} a_{2l}(s)\right). \tag{A26}$$

Expanding the product of sums in the r.h.s. of (A26)

$$G(r) = 1 - \sum_{s_1=r+1}^{\infty} \left(\sum_{l=1}^{\infty} a_{2l}(s)\right) + \sum_{s_1=r+1}^{\infty} \sum_{s_2=s_1+1}^{\infty} \left(\sum_{l=1}^{\infty} a_{2l}(s_1)\right)\left(\sum_{l=1}^{\infty} a_{2l}(s_2)\right) - \cdots, \tag{A27}$$

summing up over s_i and collecting the terms with the definite multiplicity of integration, one can represent (A27) in the form

$$G(r) = \sum_{l=0}^{\infty} g_{2l}(r). \tag{A28}$$

The coefficients $g_{2l}(r)$ can be obtained from (A27) directly but it is more simple to compute them by use the following recursion relation

$$g_{2l}(r) + \sum_{k=0}^{l-1} \sum_{s=r+1}^{\infty} g_{2k}(s) a_{2(l-k)}(s) = 0. \tag{A29}$$

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It is not difficult to deduce this relation from (A27). Using the eq. (A18) for $a_{2l}(s)$ and initial condition $g_0(s) = 1$ one can obtain from (A29)

$$g_{2l}(r) = \frac{(-1)^l}{l!l!(2\pi i)^{2l}} \oint_{|z_i|<1} \frac{\prod_{i=1}^{2l}(dz_i\, z_i^r) \prod_{i=1}^{l-1}\prod_{j=i+1}^{l}[(z_{2i-1}-z_{2j-1})^2(z_{2i}-z_{2j})^2] \prod_{i=1}^{l} W(z_{2i-1})}{\prod_{i=1}^{l}\prod_{j=1}^{l}(1-z_{2i-1}z_{2j})^2 \quad \prod_{i=1}^{l} W(z_{2i}^{-1})}, \quad \text{(A30)}$$

in particular,

$$g_2(r) = -\frac{1}{(2\pi i)^2} \oint_{|z_i|<1} \frac{d(z_1 z_2)\,(z_1 z_2)^r}{(1-z_1 z_2)^2} \frac{W(z_1)}{W(z_2^{-1})},$$

$$g_4(r) = -\frac{1}{4(2\pi i)^4} \oint_{|z_i|<1} \frac{d(z_1 z_2 z_3 z_4)\,(z_1 z_2 z_3 z_4)^r (z_1-z_3)^2(z_2-z_4)^2}{(1-z_1 z_2)^2(1-z_2 z_3)^2(1-z_3 z_4)^2(1-z_4 z_1)^2} \frac{W(z_1)W(z_3)}{W(z_2^{-1})W(z_4^{-1})}.$$

Note that the integrand of (A30) is symmetric both with respect to permutations of the even subscript variables and with respect to permutations odd subscript variables.

The structure of Toeplitz matrix is altered in paramagnetic region. Here it is necessary to evaluate the determinant of the matrix which has the following form

$$B^{(r)} = \begin{pmatrix} A_{-1} & A_0 & A_1 & \cdots & A_{r-2} \\ A_{-2} & A_{-1} & A_0 & \cdots & A_{r-3} \\ A_{-3} & A_{-2} & A_{-1} & \cdots & A_{r-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{-r} & A_{-r+1} & A_{-r+2} & \cdots & A_{-1} \end{pmatrix}, \quad \text{(A31)}$$

$$B_{x,x'} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} z^{-x+x'-1} A(z).$$

The technique stated above is not valid because the kernel of the matrix $B^{(r)}$

$$B(z) = z^{-1} A(z)$$

does not satisfy the Loran expansion condition for $\ln B(z)$ if $\ln A(z)$ does satisfy. Nevertheless, the problem can be reduced to that considered above. Really, it follows from the definition of the inverse matrix element through the determinant and cofactor

$$\det B^{(r)} = (-1)^r (A^{(r+1)})^{-1}_{r,0} \det A^{(r+1)}.$$

Using the eqs. (A15), (A16) for the inverse matrix and the eqs. (A25), (A28), (A30) for the determinant one can obtain

$$\det B^{(r)} = (-1)^r e^{h(r+1)} F(r), \quad \text{(A32)}$$

where

$$F(r) = G(r + 1) \cdot (A^{(r+1)})_{r,0}^{-1} = \frac{1}{Q^2(0)} \sum_{l=0}^{\infty} f_{2l+1}(r), \tag{A33}$$

$$f_{2l+1}(r) = \sum_{k=0}^{l} a_{2k+1}(r + 1)g_{2(l-k)}(r + 1). \tag{A34}$$

All summands in the r.h.s. of (A34) are the integrals of $2l + 1$ multiplicity. After reduction the integrands to common denominator and symmetrization over even subscript variables and over odd subscript variables one can obtain

$$f_{2l+1}(r) = \frac{(-1)^l}{l!(l+1)!(2\pi i)^{2l+1}} \oint_{|z_i|<1} \prod_{i=1}^{2l+1} (dz_i\, z_i^r) \times$$

$$\times \frac{\prod_{i=1}^{l-1}\prod_{j=i+1}^{l}(z_{2i} - z_{2j})^2 \prod_{i=1}^{l}\prod_{j=i+1}^{l+1}(z_{2i-1} - z_{2j-1})^2}{\prod_{i=1}^{l}\prod_{j=0}^{l}(1 - z_{2i}z_{2j+1})^2} \frac{\prod_{i=1}^{l}(z_{2i}W(z_{2i}))}{\prod_{i=0}^{l}(z_{2i+1}W(z_{2i+1}^{-1}))}, \tag{A35}$$

and, in particular,

$$f_1(r) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz\, z^r}{zW(z^{-1})},$$

$$f_3(r) = -\frac{1}{2(2\pi i)^3} \oint_{|z_i|<1} \frac{d(z_1z_2z_3)\,(z_1z_2z_3)^r (z_1 - z_3)^2}{(1 - z_1z_2)^2(1 - z_2z_3)^2(1 - z_3z_1)^2} \frac{z_2W(z_2)}{z_1z_3W(z_1^{-1})W(z_3^{-1})}.$$

References

[1] *E.W. Montroll, R.A. Potts and J.C. Ward.* Correlations and spontaneous magnetization of the two-dimensional Ising model. J. Math. Phys. 1963. <u>4</u>. P. 308–322.

[2] *B.M. McCoy.* The connection between statistical mechanics and quantum field theory. In: Statistical Mechanics and Field Theory. Eds. V.V. Barhanov, C.J. Burden. World Scientific. 1995. P. 26–128.

[3] *B.M. McCoy, T.T. Wu.* The Two-Dimensional Ising Model. Cambridge: Harvard University Press, 1973.

[4] *T.T. Wu, B.M. McCoy, C.A. Tracy and E. Barouch.* Spin-spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region. Phys. Rev. B. 1976. V. 13. P. 316–374.

[5] *C.R. Nappi.* On the scaling limit of the Ising model. Nuovo Cimento A. 1978. V. 44. P. 392–400.

[6] *J. Palmer and C.A. Tracy.* Two-dimensional Ising correlations: convergence of the scaling limit. Advances in Applied Mathematics. 1981. V.2. P. 329–388.

[7] *B. Berg, M. Karowski and P. Weisz.* Construction of Green's functions from an exact S matrix. Phys. Rev. D. 1979. V.19. P. 2477-2479.

[8] *A.B. Zamolodchikov, Al.B. Zamolodchikov.* Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models. Ann. Phys. 1979. V. 120. P. 253.

[9] *F.A. Smirnov.* Form factors in completely integrable models of quantum field theory. Adv. Series in Math. Phys. 14. Singapore: World Scientific, 1992.

[10] *S. Sachdev.* Universal, finite temperature, crossover functions of the quantum transition in the Ising chain in a transverse field. Preprint cond-mat 95091147, 21 p. (1995); Nucl.Phys. B **464**, P. 576 (1996).

[11] *A.G. Izergin, N.A. Kitanin, N.A. Slavnov.* About correlation functions of the XY model. Proceedings of the PDMI seminar, 1995. V. 226. P. 178.

[12] *A. Leclair, F. Lesage, S. Shachdev and H. Saleur.* Finite temperature correlations in the one-dimensional quantum Ising model. Nucl. Phys. B. 1996. V. 482. P. 579–602.

[13] *F.A. Smirnov.* Quasi-classical study of form factors in finite volume. 1998. Preprint LPTHE-98-10. (Paris U., VI—VII). P. 1–21. hep-th/9802132.

[14] *A. Leclair, G. Mussardo.* Finite Temperature correlation functions in integrable QFT. Nucl. Phys. B. 1999. V. 552. P. 624–642.

[15] *H. Saleur.* A comment on finite temperature correlations in integrable QFT. Nucl. Phys. B. 2000. V. 567. P. 602–610.

[16] *S. Lukyanov.* Finite temperature expectation values of local fields in sinh-Gordon model. 2000. Preprint (RUHNETC–2000–17). hep-th/0005027.

[17] *V.N. Plechko.* A simple solution of the two-dimensional Ising model on a torus via grassmann integrals. Teor. Mat. Fiz. 1985. **64**. P. 150–162.

[18] *E.A. Bugrii.* Solution of the $2D$ Ising model on a triangular lattice by the method of auxiliary q-deformed grassmann fields. Theoretical and Mathematical Physics, 1996. **109**. No. 3. P. 1590–1607.

[19] *A.I. Bugrij, V.N. Shadura.* Duality relation for the two-dimensional Ising model at finite lattice dimensions. JETP **82** (3), March 1996. P. 552–558.

[20] *I.S. Gradshtein, I.M. Ryzhik.* The tables of integral, sums and products. – M.: "Nauka", 1971.